

# On the Bousfield-Kan spectral sequence for $Q(2)_{(3)}$

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## Abstract

We compute the  $E_2$ -term of the Bousfield-Kan spectral sequence converging to the homotopy groups of the semi-cosimplicial  $E_\infty$  ring spectrum  $Q(2)_{(3)}$ . This 3-local spectrum was constructed by M. Behrens using degree 2 isogenies of elliptic curves, and its localization with respect to the 2nd Morava  $K$ -theory  $K(2)$  is “one half” of the  $K(2)$ -local sphere at the prime 3. The computation in this paper uses techniques developed in the author’s previous work [9] on the Adams-Novikov spectral sequence for  $Q(2)_{(3)}$ , and provides another gateway to the homotopy ring  $\pi_*Q(2)_{(3)}$ .

## 1 Introduction

In his work on the  $K(2)$ -local sphere, Behrens ([2, 3]) constructs a  $p$ -local  $E_\infty$  ring spectrum  $Q(N)_{(p)}$  for each prime  $p$  and positive integer  $N$  not divisible by  $p$ . For fixed  $p$  and  $N$ , this spectrum is the homotopy limit of a semi-cosimplicial diagram  $Q(N)_{(p)}^\bullet$  of the form

$$Q(N)_{(p)}^\bullet : \quad TMF_{(p)} \rightrightarrows TMF_0(N)_{(p)} \vee TMF_{(p)} \rightrightarrows TMF_0(N)_{(p)} \quad (1)$$

where  $TMF_{(p)}$  and  $TMF_0(N)_{(p)}$  are the  $p$ -localizations of topological modular forms and its analog for  $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$ , respectively. The arrows in (1) denote alternating sums of two (resp. three) coface maps, each defined in terms of degree  $N$  isogenies of elliptic curves.

In the case  $p = 3$  and  $N = 2$ , the  $K(2)$ -localization of (1) is a reinterpretation of the  $K(2)$ -local sphere resolution constructed by Goerss, Henn, Mahowald, and Rezk [7]. Moreover, Behrens ([2], Theorem 2.0.1) has shown that

$$DL_{K(2)}Q(2)_{(3)} \xrightarrow{DL_{K(2)}\eta} L_{K(2)}S \xrightarrow{L_{K(2)}\eta} L_{K(2)}Q(2)_{(3)} \quad (2)$$

is a cofiber sequence, where  $L_{K(2)}$  is Bousfield localization with respect to  $K(2)$ ,  $\eta$  is the unit map of  $Q(2)_{(3)}$ , and  $D$  is the  $K(2)$ -local Spanier-Whitehead duality functor. Therefore, the  $K(2)$ -local sphere decomposes in terms of  $L_{K(2)}Q(2)_{(3)}$  and  $DL_{K(2)}Q(2)_{(3)}$ , and the homotopy groups of these three spectra are intertwined. The analog of (2) for  $p = 5$  and  $N = 2$  is

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known to be a cofiber sequence, and is conjectured to be so for all primes  $p$  and corresponding  $N$  ([3], Conjecture 1.6.1). The case  $p = 2$  is addressed by Behrens and K. Ormsby in [5].

In this paper, we leverage the cosimplicial structure of  $Q(2)_{(3)}$  and compute the  $E_2$ -term of the Bousfield-Kan spectral sequence (BKSS) converging to  $\pi_*Q(2)_{(3)}$ . We shall denote the  $E_2$ -term of this spectral sequence by  ${}_{BK}E_2^{s,t}Q(2)_{(3)}$ . Our computation gives explicit descriptions of the elements in this  $E_2$ -term up to an ambiguity in a torsion  $\mathbb{Z}_{(3)}$ -module which we denote  $U^* \subset {}_{BK}E_2^{1,*}Q(2)_{(3)}$ . Throughout this paper,  $\nu_3(x)$  will denote the 3-adic valuation of a (3-local) integer  $x$ . The following is our main theorem.

**Theorem 1.1.** *The Bousfield-Kan  $E_2$ -term for  $Q(2)_{(3)}$  is given by*

$${}_{BK}E_2^{0,t}Q(2)_{(3)} = \begin{cases} \mathbb{Z}_{(3)}, & t = 0, \\ (\pi_t T M F_{(3)})_T, & t \neq 0, \end{cases}$$

${}_{BK}E_2^{1,t}Q(2)_{(3)} = (\pi_t T M F_{(3)})_T \oplus M^1$  where

$$M^1 = \begin{cases} \bigoplus_{n \in \mathbb{N}} (\mathbb{Z}_{(3)} \oplus \mathbb{Z}/(3)), & t = 0, \\ \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/(3), & t = 4m, m < 0, \\ \mathbb{Z}/(3^{\nu_3(3m)}) \oplus \left( \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/(3) \right), & t = 4m, m > 0, \\ \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/(3), & t = 4m + 2, m \leq 0, \\ U^t \oplus \left( \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/(3) \right), & t = 4m + 2, m \geq 1, m \equiv 13 \pmod{27}, \\ \mathbb{Z}/(3^{\nu_3(6m+3)}) \oplus \left( \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/(3) \right), & t = 4m + 2, m \geq 1, m \not\equiv 13 \pmod{27}, \\ 0, & \text{otherwise,} \end{cases}$$

$${}_{BK}E_2^{2,t}Q(2)_{(3)} = \begin{cases} \bigoplus_{n \in \mathbb{N}} (\mathbb{Z}_{(3)} \oplus \mathbb{Z}/(3)), & t = 0, \\ \bigoplus_{n \in \mathbb{N}} (\mathbb{Z}/(3) \oplus \mathbb{Z}/(3^{\nu_3(3m)})), & t = 4m, m \neq 0, \\ \bigoplus_{n \in \mathbb{N}} (\mathbb{Z}/(3) \oplus \mathbb{Z}/(3^{\nu_3(6m+3)})), & t = 4m + 2, m \leq 0, \\ \left( \bigoplus_{n \in \mathbb{N}} (\mathbb{Z}/(3) \oplus \mathbb{Z}/(3^{\nu_3(6m+3)})) \right) / \sim, & t = 4m + 2, m \geq 1, m \equiv 13 \pmod{27}, \\ \bigoplus_{n \in \mathbb{N}} (\mathbb{Z}/(3) \oplus \mathbb{Z}/(3^{\nu_3(6m+3)})), & t = 4m + 2, m \geq 1, m \not\equiv 13 \pmod{27}, \\ 0, & \text{otherwise} \end{cases}$$

where  $\sim$  denotes a single relation among the generators, and  ${}_{BK}E_2^{s,t}Q(2)_{(3)} = 0$  for  $s \geq 3$ .

In Section 2 we list the results that comprise our proof of Theorem 1.1. In Section 3 we set up the BKSS for  $Q(2)_{(3)}$ , along with several algebraic tools needed for pursuing the

$E_2$ -term. Sections 4 and 5 are the most technical, and contain the computations needed to finish the proof of Theorem 1.1. Finally, in Section 6 we briefly examine the structure of the differentials on the  $E_2$ -page and beyond in the BKSS.

## 2 Statement of main results

This section outlines our strategy for proving Theorem 1.1. Our approach resembles that from our previous work [9] on the Adams-Novikov  $E_2$ -term for  $Q(2)_{(3)}$ . In Propositions 2.1 – 2.3 below, there are references to two graded  $\mathbb{Z}_{(3)}$ -algebras,  $B$  and  $\Gamma$ ; these algebras piece together to form an elliptic curve Hopf algebroid that will be defined in Section 3. This Hopf algebroid is the key algebraic object underlying our computation.

To begin, we express the Bousfield-Kan  $E_2$ -term for  $Q(2)_{(3)}$  algebraically.

**Proposition 2.1.** *The BKSS  $E_2$ -term for  $Q(2)_{(3)}$  is the cohomology of a semi-cosimplicial abelian group. More precisely, it has the form*

$${}_{BK}E_2^{s,t}Q(2)_{(3)} = H^s(\pi_t TMF_{(3)} \Rightarrow B_{t/2} \times \pi_t TMF_{(3)} \Rrightarrow B_{t/2})$$

where the coface maps are induced by the corresponding maps in (1).

For notational convenience, we put

$$\mathcal{G}_*^\bullet := (\pi_* TMF_{(3)} \Rightarrow B_{*/2} \times \pi_* TMF_{(3)} \Rrightarrow B_{*/2}) \quad (3)$$

so that  ${}_{BK}E_2^{s,t}Q(2)_{(3)} = H^s \mathcal{G}_t^\bullet$  by Proposition 2.1. The following proposition describes a two-stage filtration that we use to compute  $H^* \mathcal{G}_*^\bullet$ .

**Proposition 2.2.** *There is a filtration  $\mathcal{G}_*^\bullet = F^0 \supset F^1 \supset F^2$  of  $\mathcal{G}_*^\bullet$  inducing a short exact sequence  $0 \rightarrow C' \rightarrow \mathcal{G}_*^\bullet \rightarrow C'' \rightarrow 0$ , where*

$$C' = (0 \rightarrow B_{*/2} \xrightarrow{h} B_{*/2}), \quad C'' = (\pi_* TMF_{(3)} \xrightarrow{g} \pi_* TMF_{(3)} \rightarrow 0).$$

The resulting long exact sequence in cohomology is

$$0 \rightarrow H^0 \mathcal{G}_*^\bullet \rightarrow \ker g \xrightarrow{\delta^0} \ker h \rightarrow H^1 \mathcal{G}_*^\bullet \rightarrow \operatorname{coker} g \xrightarrow{\delta^1} \operatorname{coker} h \rightarrow H^2 \mathcal{G}_*^\bullet \rightarrow 0$$

so that  $H^0 \mathcal{G}_*^\bullet = \ker \delta^0$ ,  $H^2 \mathcal{G}_*^\bullet = \operatorname{coker} \delta^1$ , and  $H^1 \mathcal{G}_*^\bullet$  lies in the short exact sequence

$$0 \rightarrow \operatorname{coker} \delta^0 \rightarrow H^1 \mathcal{G}_*^\bullet \rightarrow \ker \delta^1 \rightarrow 0. \quad (4)$$

Moreover,  $\operatorname{coker} \delta^0$  is concentrated in  $t$ -degree 0 and  $\ker \delta^1$  is free of rank 1 in  $t$ -degree 0, so that the sequence (4) splits in  $t$ -degree 0 and  $H^1 \mathcal{G}_*^\bullet \cong \ker \delta^1$  in nonzero  $t$ -degrees.

Leveraging the filtration given in Proposition 2.2 requires computing the kernels and cokernels of the maps  $g$  and  $h$ . We do so by applying some technical results from our previous work [9] on the ANSS for  $Q(2)_{(3)}$ . Throughout this paper,  $A_T$  will denote the torsion subgroup of an abelian group  $A$ .

**Proposition 2.3.** As  $\mathbb{Z}_{(3)}$ -modules,

$$\begin{aligned}\ker g &= \pi_0 TMF_{(3)} \oplus (\pi_* TMF_{(3)})_T, \\ \ker h &= \bigoplus_{n \in \mathbb{N}} \mathbb{Z}_{(3)},\end{aligned}$$

and

$$\begin{aligned}\text{coker } g &= \pi_* TMF_{(3)} / \sim_{\text{coker } g}, \\ \text{coker } h &= \left( \bigoplus_{n \in \mathbb{N}} \mathbb{Z}_{(3)} \right) \oplus \left( \bigoplus_{\substack{i < j \in \mathbb{Z} \\ i+j \neq 0}} \mathbb{Z} / (3^{\nu_3(i+j)+1}) \right) \oplus \left( \bigoplus_{i < j \in \mathbb{Z}} \mathbb{Z} / (3^{\nu_3(2i+2j+1)+1}) \right)\end{aligned}$$

where  $\sim_{\text{coker } g}$  denotes the set of all relations of the form  $3^{\nu_3(t)+1} \cdot x = 0$  for  $x \in \pi_* TMF_{(3)}$  represented by an element in  $\text{Ext}_{\Gamma}^{0,t}(B, B)$  with  $t \neq 0$ .

The following theorem describes  $H^* \mathcal{G}_*^\bullet$ . Using results from [9] once again, we prove this result by computing the connecting homomorphisms  $\delta^0$  and  $\delta^1$  from Proposition 2.2.

**Theorem 2.4.** (a)  $H^0 \mathcal{G}_*^\bullet = \mathbb{Z}_{(3)} \oplus (\pi_* TMF_{(3)})_T$

$$\begin{aligned}(b) \quad H^1 \mathcal{G}_*^\bullet &= \left( \bigoplus_{n \in \mathbb{N}} \mathbb{Z}_{(3)} \right) \oplus \left( \bigoplus_{n \in \mathbb{N}} \mathbb{Z} / (3) \right) \oplus (\pi_* TMF_{(3)})_T \oplus \left( \bigoplus_{m > 0} \mathbb{Z} / (3^{\nu_3(3m)}) \right) \\ &\quad \oplus \left( \bigoplus_{\substack{m > 0 \\ m \neq 13 \pmod{27}}} \mathbb{Z} / (3^{\nu_3(6m+3)}) \right) \oplus U^*\end{aligned}$$

$$\begin{aligned}(c) \quad H^2 \mathcal{G}_*^\bullet &= \left( \bigoplus_{n \in \mathbb{N}} \mathbb{Z}_{(3)} \right) \oplus \left( \bigoplus_{n \in \mathbb{N}} \mathbb{Z} / (3) \right) \oplus \left( \bigoplus_{m \neq 0} \bigoplus_{n \in \mathbb{N}} \mathbb{Z} / (3^{\nu_3(3m)}) \right) \oplus \left( \bigoplus_{m \leq 0} \bigoplus_{n \in \mathbb{N}} \mathbb{Z} / (3^{\nu_3(6m+3)}) \right) \\ &\quad \oplus \left( \bigoplus_{\substack{m > 0 \\ m \neq 13 \pmod{27}}} \bigoplus_{n \in \mathbb{N}} \mathbb{Z} / (3^{\nu_3(6m+3)}) \right) \oplus \left( \bigoplus_{\substack{m > 0 \\ m \equiv 13 \pmod{27}}} \left( \bigoplus_{n \in \mathbb{N}} (\mathbb{Z} / (3) \oplus \mathbb{Z} / (3^{\nu_3(6m+3)})) \right) \right) / \sim\end{aligned}$$

*Proof of Theorem 1.1.* Our proof of Theorem 2.4 will reveal that the generators of  $\mathbb{Z}_{(3)}$  summands lie in  ${}_{BK}E_2^{*,0}Q(2)_{(3)}$ , the generators of  $\mathbb{Z} / (3^{\nu_3(3m)})$  (resp.  $\mathbb{Z} / (3^{\nu_3(6m+3)})$ ) lie in  ${}_{BK}E_2^{*,4m}Q(2)_{(3)}$  (resp.  ${}_{BK}E_2^{*,4m+2}Q(2)_{(3)}$ ), and the direct sums of countably many copies of  $\mathbb{Z} / (3)$  lie in the bidegrees Theorem 1.1 indicates. The relation  $\sim$  appearing in Theorem 1.1 and Theorem 2.4(c) is defined below in Lemma 5.8(a). Coupled with these facts, Propositions 2.1 and 2.2 and Theorem 2.4 together prove Theorem 1.1.

### 3 $Q(2)_{(3)}$ and its Bousfield-Kan spectral sequence

In this section we set up the Bousfield-Kan spectral sequence for  $Q(2)_{(3)}$  and its underlying algebraic apparatus. In particular, we prove Propositions 2.1 and 2.2.

### 3.1 The homotopy of $TMF_{(3)}$ and $TMF_0(2)_{(3)}$

We begin by formally defining  $Q(2)_{(3)}$  in terms of the semi-cosimplicial diagram (1) with  $p = 3$ ,  $N = 2$ . The coface maps will be defined later in this section in terms of algebraic data.

**Definition 1.** The spectrum  $Q(2)_{(3)}$  is given by

$$Q(2)_{(3)} := \operatorname{holim}(TMF_{(3)} \Rightarrow TMF_0(2)_{(3)} \vee TMF_{(3)} \Rrightarrow TMF_0(2)_{(3)}),$$

that is,  $Q(2)_{(3)} = \operatorname{holim} Q(2)_{(3)}^\bullet$ .

The homotopy rings  $\pi_* TMF_{(3)}$  and  $\pi_* TMF_0(2)_{(3)}$  will play a key role in our computation, and they both arise from the data in an elliptic curve Hopf algebroid over  $\mathbb{Z}_{(3)}$  that we shall denote  $(B_*, \Gamma_*)$  (we will often suppress the grading and simply write  $(B, \Gamma)$ ). This Hopf algebroid co-represents the groupoid of non-singular elliptic curves over  $\mathbb{Z}_{(3)}$  with Weierstrass equation

$$y^2 = 4x(x^2 + q_2x + q_4) \quad (5)$$

and isomorphisms  $x \mapsto x + r$  that preserve this Weierstrass form.

**Definition 2.** As graded  $\mathbb{Z}_{(3)}$ -algebras,

$$\begin{aligned} B_* &:= \mathbb{Z}_{(3)}[q_2, q_4, \Delta^{-1}] / (\Delta = q_4^2(16q_2^2 - 64q_4)), \\ \Gamma_* &:= B_*[r] / (r^3 + q_2r^2 + q_4r) \end{aligned}$$

where  $q_2 \in B_2$ ,  $q_4 \in B_4$  (hence  $\Delta \in B_{12}$ ), and  $r \in \Gamma_2$ .

The following lemma gives Adams-Novikov-style spectral sequences converging to  $\pi_* TMF_{(3)}$  and  $\pi_* TMF_0(2)_{(3)}$ , respectively (see, e.g., [2], Corollary 1.4.2).

**Lemma 3.1.** (a) *The Adams-Novikov spectral sequence (ANSS) converging to  $\pi_* TMF_{(3)}$  has the form*

$${}_{AN}E_2^{s,t} TMF_{(3)} := \operatorname{Ext}_\Gamma^{s,t}(B, B) \Rightarrow \pi_{2t-s} TMF_{(3)} \quad (6)$$

where  $\operatorname{Ext}_\Gamma^{*,*}(B, B)$  is the cohomology of  $(B, \Gamma)$ .

(b) *The ANSS converging to  $\pi_* TMF_0(2)_{(3)}$  has the form*

$${}_{AN}E_2^{s,t} TMF_0(2)_{(3)} := \operatorname{Ext}_B^{s,t}(B, B) \Rightarrow \pi_{2t-s} TMF_0(2)_{(3)}$$

and therefore collapses at the  $E_2$ -page, yielding

$$\pi_{2k} TMF_0(2)_{(3)} = B_k$$

for all  $k \in \mathbb{Z}$ . In particular,  $\pi_* TMF_0(2)_{(3)}$  is concentrated in dimensions congruent to 0 modulo 4.

The spectral sequence (6) has been computed by Hopkins and Miller. Expository accounts of this computation can be found in [1], [8], and [10]. We recall the results of this computation here.

**Lemma 3.2.** *The homotopy ring of  $TMF_{(3)}$  is*

$$\pi_* TMF_{(3)} = \mathbb{Z}_{(3)}[c_4, c_6, 3\Delta, 3\Delta^2, \Delta^3, \Delta^{-3}, \alpha, \beta, b] / \sim_{TMF}$$

where  $c_4 \in \pi_8$ ,  $c_6 \in \pi_{12}$ ,  $3\Delta \in \pi_{24}$ ,  $\alpha \in \pi_3$ ,  $\beta \in \pi_{10}$ ,  $b \in \pi_{27}$ , and

$$\sim_{TMF} = \begin{cases} c_4^3 - c_6^2 = 576 \cdot 3\Delta, \\ 3\alpha = 3\beta = 3b = 0, \\ \alpha \cdot 3\Delta = \alpha \cdot 3\Delta^2 = \beta \cdot 3\Delta = \beta \cdot 3\Delta^2 = 0, \\ \alpha^2 = \alpha\beta^2 = \beta^5 = 0, \\ c_4\alpha = c_6\alpha = c_4\beta = c_6\beta = c_4b = c_6b = 0. \end{cases}$$

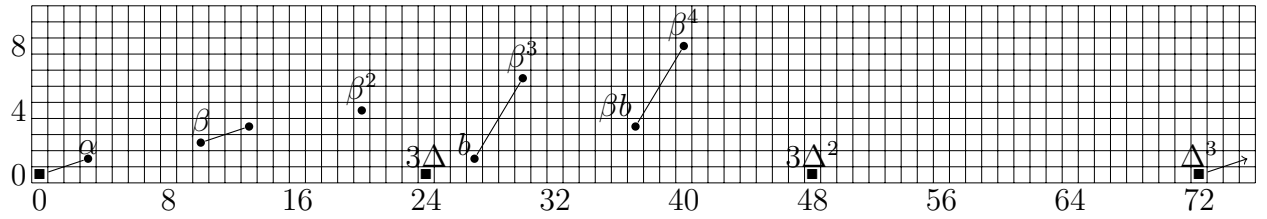


Figure 1: The  $E_\infty$ -term of the ANSS for  $TMF_{(3)}$ ,  $0 \leq 2t - s \leq 75$ .

The  $E_\infty$ -term of (6) is shown (with some elements omitted) in Figure 1, with  $s$  along the vertical axis and  $2t - s$  along the horizontal axis. A black square denotes a copy of  $\mathbb{Z}_{(3)}$  and a black circle denotes a copy of  $\mathbb{Z}/(3)$ . Lines of slope  $1/3$  represent multiplication by  $\alpha$ , and lines of slope  $5/3$  represent multiplicative extensions induced by the relation  $b\alpha = \beta^3$  in  $\pi_{27}$ . Not shown in the chart are most of the elements on the line  $s = 0$ ; these elements will be enumerated below in Subsection 3.4. The portion of the  $E_\infty$ -term shown is 72-periodic with invertible generator  $\Delta^3$ .

**Remark 1.** The groups  $\text{Ext}_\Gamma^{*,*}(B, B)$  are encoded as the cohomology groups of the *cobar resolution* for  $(B, \Gamma)$ , a cochain complex of the form

$$B \rightarrow \Gamma \rightarrow \Gamma \otimes \Gamma \rightarrow \Gamma \otimes \Gamma \otimes \Gamma \rightarrow \dots$$

([11], A1.2.11) where the differentials are defined in terms of the structure maps of  $(B, \Gamma)$ . For example,  $\alpha \in \pi_3 TMF_{(3)}$  is represented by  $r \in \Gamma$  in the cobar resolution; similarly,  $b \in \pi_{27} TMF_{(3)}$  is represented by  $r\Delta \in \Gamma$ , and  $\beta \in \pi_{10} TMF_{(3)}$  is represented by  $r^2 \otimes r - r \otimes r^2 \in \Gamma \otimes \Gamma$ . Representatives for the other torsion elements in  $\pi_* TMF_{(3)}$  can be computed via the cobar resolution pairing defined in ([11], A1.2.15).

### 3.2 Coface maps and the BKSS setup

To define the coface maps appearing in Definition 1, we will need four Hopf algebroid maps:

$$\begin{aligned}\psi_d &: (B, B) \rightarrow (B, B), \\ \phi_f &: (B, \Gamma) \rightarrow (B, B), \\ \phi_q &: (B, \Gamma) \rightarrow (B, B), \\ \psi_{[2]} &: (B, \Gamma) \rightarrow (B, \Gamma).\end{aligned}$$

**Definition 3.** The maps  $\psi_d$ ,  $\phi_f$ ,  $\phi_q$ , and  $\psi_{[2]}$  are induced by the following maps of  $\mathbb{Z}_{(3)}$ -algebras:

$\psi_d : B \rightarrow B$	$\phi_f : \Gamma \rightarrow B$	$\phi_q : \Gamma \rightarrow B$	$\psi_{[2]} : \Gamma \rightarrow \Gamma$
$q_2 \mapsto -2q_2,$	$q_2 \mapsto q_2,$	$q_2 \mapsto -2q_2,$	$q_2 \mapsto 4q_2,$
$q_4 \mapsto q_2^2 - 4q_4.$	$q_4 \mapsto q_4,$	$q_4 \mapsto q_2^2 - 4q_4,$	$q_4 \mapsto 16q_4,$
	$r \mapsto 0.$	$r \mapsto 0.$	$r \mapsto 4r.$

**Remark 2.** These maps are defined in terms of certain maneuvers with Weierstrass equations of elliptic curves, and are therefore most naturally defined in the context of algebraic stacks. We shall not discuss these maneuvers here, but the details can be found in ([2], Section 1.5) and are recapitulated in ([9], Section 3.3).

We may now define the coface maps. We do so on the level of Hopf algebroids, and the maps in Definition 1 are the corresponding maps of spectra (which, by abuse of notation, will be denoted using the same symbols).

**Definition 4.** The map out of  $TMF_{(3)}$  is induced by the alternating sum  $d_0 - d_1$ , where

$$d_0 := \phi_q \oplus \psi_{[2]}, \quad d_1 := \phi_f \oplus 1_{(B, \Gamma)},$$

and the map into  $TMF_0(2)_{(3)}$  is induced by the alternating sum  $\overline{d}_0 - \overline{d}_1 + \overline{d}_2$ , where

$$\overline{d}_0 := \psi_d, \quad \overline{d}_1 := \phi_f, \quad \overline{d}_2 := 1_{(B, B)}.$$

To elucidate the structure of the BKSS for  $Q(2)_{(3)}$ , we start with an observation of Behrens [4] about the  $E_1$ -term, namely that

$${}_{BK}E_1^{s,t}Q(2)_{(3)} = \pi_t Q(2)_{(3)}^s \Rightarrow \pi_{t-s} Q(2)_{(3)}.$$

In other words, the  $E_1$ -term is the semi-cosimplicial abelian group  $\pi_* Q(2)_{(3)}^\bullet = \mathcal{G}_*^\bullet$ . The  $E_2$ -term is therefore obtained by taking cohomology of  $\mathcal{G}_*^\bullet$  at each of its three nontrivial stages. Within the general framework of the Bousfield-Kan technology, the relevant notion is that of cohomotopy.

**Definition 5.** The  $s$ th cohomotopy group  $\pi^s G^\bullet$  of a (semi-)cosimplicial abelian group  $G^\bullet$  is given by

$$\pi^s G^\bullet := H^s G^\bullet$$

where cohomology is taken with respect to the alternating sums of the coface maps of  $G^\bullet$ .

Therefore,

$${}_{BK}E_2^{s,t}Q(2)_{(3)} = \pi^s \pi_t Q(2)_{(3)}^\bullet = H^s \pi_t Q(2)_{(3)}^\bullet$$

and since  $\pi_t TMF_0(2)_{(3)} = B_{t/2}$  by Lemma 3.1(b), this proves Proposition 2.1.

### 3.3 Filtration of $\mathcal{G}_\bullet^\bullet$

We now define the filtration  $\mathcal{G}_\bullet^\bullet = F^0 \supset F^1 \supset F^2$  asserted to exist in Proposition 2.2. For ease of notation, we will henceforth denote by 1 the maps  $1_{(B,B)}$ ,  $1_{(B,\Gamma)}$ , and any maps they induce or correspond to; the meaning will be clear from the context.

The filtration we want is  $F^1 = (\pi_* TMF_{(3)} \xrightarrow{\psi_{[2]}-1} \pi_* TMF_{(3)} \rightarrow 0)$  and  $F^2$  is the trivial complex. The induced short exact sequence is given by

$$\begin{array}{ccccccc}
 C' : & & 0 & \xrightarrow{\quad} & B & \xrightarrow{\psi_d+1} & B \\
 & & \downarrow & & \downarrow & & \parallel \\
 \mathcal{G}_\bullet^\bullet : & & \pi_* TMF_{(3)} & \xrightarrow{\quad} & B \oplus \pi_* TMF_{(3)} & \xrightarrow{\quad} & B \\
 & & \parallel & & \downarrow & & \downarrow \\
 C'' : & & \pi_* TMF_{(3)} & \xrightarrow{\psi_{[2]}-1} & \pi_* TMF_{(3)} & \xrightarrow{\quad} & 0
 \end{array}$$

which leads to the following definition of the maps  $g$  and  $h$  appearing in the statement of Proposition 2.2.

**Definition 6.**  $g := \psi_{[2]} - 1 : \pi_* TMF_{(3)} \rightarrow \pi_* TMF_{(3)}$ ,  $h := \psi_d + 1 : B \rightarrow B$ .

With the exception of the splitting of the short exact sequence (4), Proposition 2.2 follows from standard homological algebra (see, e.g., Section 1.3 of [12]). The connecting map  $\delta^0$  is the restriction of  $\phi_q - \phi_f$  to  $\ker g$ , while the connecting map  $\delta^1$  is induced by  $-\phi_f$ .

### 3.4 Algebraic properties of $B_*$ and $\pi_* TMF_{(3)}$

In this subsection we lay the algebraic groundwork for our computation by examining the rings  $B_*$  and  $\pi_* TMF_{(3)}$ . The results and notation from this subsection parallel those in ([9], Subsection 3.2).

We start with  $B_*$ , where we define a new elements  $\sigma, \tau \in B_4$  as follows (cf. [9], Eq. (10)):

$$\sigma := 8q_4, \quad \tau := \frac{16q_2^2 - 64q_4}{8}.$$

**Definition 7.** For  $i < j \in \mathbb{Z}$ ,  $a_{i,j} := \sigma^i \tau^j - \sigma^j \tau^i$  and  $b_{i,j} := a_{i,j} q_2$ .

The elements  $\{a_{i,j}\}$  and  $\{b_{i,j}\}$  from Definition 7 are collectively a subset of a basis of eigenvectors for  $B$  with respect to the map  $h$ ; this was proven in ([9], Subsection 4.3). The following definition gives a convenient enumeration of these elements, particularly for our study of the connecting map  $\delta^1$  in Subsection 5.2.

**Definition 8.** For  $0 \leq v \in \mathbb{Z}$  and  $m \in \mathbb{Z}$ ,

$$A_v^m := a_{\lfloor \frac{m-1}{2} \rfloor - v, \lfloor \frac{m+1}{2} \rfloor + v}, \quad B_v^m := b_{\lfloor \frac{m-1}{2} \rfloor - v, \lfloor \frac{m+1}{2} \rfloor + v}.$$

The following lemma is a restatement of ([9], Lemma 3).



**Lemma 3.3.**  $B = \bigoplus_{v \geq 0, m \in \mathbb{Z}} \mathbb{Z}_{(3)} \{A_v^m, B_v^m\}$

Next we record some results on  $\pi_* TMF_{(3)}$ . Specifically, we will focus on elements in  ${}_{AN}E_\infty^{0,*} TMF_{(3)}$ ; that is, elements on the 0-line of the ANSS for  $TMF_{(3)}$ .

**Lemma 3.4.** *If*

$$I_{TMF} := \{(n, \epsilon, \ell_1, \ell_2, \ell_3) : 0 \leq n \in \mathbb{Z}, \epsilon \in \{0, 1\}, (\ell_1, \ell_2) \in \{(0, 0), (0, 1), (1, 0)\}, \ell_3 \in \mathbb{Z}\},$$

*then*

$${}_{AN}E_\infty^{0,*} TMF_{(3)} = \mathbb{Z}_{(3)} \{c_4^n \cdot c_6^\epsilon \cdot [3\Delta]^{\ell_1} \cdot [3\Delta^2]^{\ell_2} \cdot [\Delta^3]^{\ell_3} : (n, \epsilon, \ell_1, \ell_2, \ell_3) \in I_{TMF}\}.$$

Lemma 3.4 gives a basis for  ${}_{AN}E_\infty^{0,*} TMF_{(3)}$  over  $\mathbb{Z}_{(3)}$ . The following lemma gives representatives for these basis elements in  $\text{Ext}_\Gamma^{0,*}(B, B)$  so that we may compute the maps into/out of  $\pi_* TMF_{(3)}$  (or subquotients thereof) in the BKSS.

**Lemma 3.5.** (a)  $\text{Ext}_\Gamma^{0,*}(B, B) = \mathbb{Z}_{(3)}[c_4, c_6, \Delta, \Delta^{-1}]/(1728\Delta = c_4^3 - c_6^2)$

(b) *If*

$$\mathcal{B}_{TMF}^{\epsilon, m} := \{\gamma c_4^n c_6^\epsilon \Delta^\ell : n \geq 0, \epsilon \in \{0, 1\}, \ell \in \mathbb{Z}, n + \epsilon + 3\ell = m\} \subset \text{Ext}_\Gamma^{0,*}(B, B)$$

*where*

$$\gamma := \begin{cases} 1, & \text{if } \ell \equiv 0 \pmod{3}, \\ 3, & \text{otherwise,} \end{cases}$$

*then*

$$\mathcal{B}_{TMF} := \coprod_{m \in \mathbb{Z}, \epsilon \in \{0, 1\}} \mathcal{B}_{TMF}^{\epsilon, m}$$

*is a complete set of representatives for the  $\mathbb{Z}_{(3)}$ -basis of  ${}_{AN}E_\infty^{0,*} TMF_{(3)}$  given in Lemma 3.4.*

*Proof.* Part (a) is proven in [6]. Part (b) follows from Lemma 3.2.  $\square$

As we do throughout this paper, we shall not distinguish between elements of  $\mathcal{B}_{TMF}$  and the homotopy elements they represent in  ${}_{AN}E_\infty^{0,*} TMF_{(3)} \subset \pi_* TMF_{(3)}$ . In particular, we may interpret Lemma 3.5 as saying that  $\mathcal{B}_{TMF}$  is a  $\mathbb{Z}_{(3)}$ -basis for  ${}_{AN}E_\infty^{0,*} TMF_{(3)}$ .

We now give notation for the submodules of  $\pi_* TMF_{(3)}$  spanned by the sets  $\mathcal{B}_{MF}^{\epsilon, m}$  defined in Lemma 3.5(b).

**Definition 9.** Given  $\epsilon \in \{0, 1\}$  and  $m \in \mathbb{Z}$ ,

$$W^{\epsilon, m} := \mathbb{Z}_{(3)} \{\mathcal{B}_{TMF}^{\epsilon, m}\} \subset {}_{AN}E_\infty^{0,*} TMF_{(3)}.$$

**Lemma 3.6.** *As a  $\mathbb{Z}_{(3)}$ -module,  ${}_{AN}E_\infty^{0,*} TMF_{(3)} = \bigoplus_{m \in \mathbb{Z}, \epsilon \in \{0, 1\}} W^{\epsilon, m}$ .*

*Proof.* This follows from Definition 9 and the disjoint union decomposition of  $\mathcal{B}_{TMF}$  given in Lemma 3.5(b).  $\square$

For a monomial  $\gamma c_4^n c_6^\epsilon \Delta^\ell \in \mathcal{B}_{TMF}^{\epsilon, m}$ , let  $\ell^{\epsilon, m}$  be the largest possible value of  $\ell$ . Then

$$\ell^{0, m} = \left\lfloor \frac{m}{3} \right\rfloor, \quad \ell^{1, m} = \left\lfloor \frac{m-1}{3} \right\rfloor.$$

Using these quantities, we can enumerate the elements in  $\mathcal{B}_{TMF}$  in a way that is convenient for our study of  $\delta^1$  in Subsection 5.2.

**Definition 10.** For  $0 \leq v \in \mathbb{Z}$  and  $m \in \mathbb{Z}$ ,

$$C_v^m := c_4^{m-3\ell^{0, m}+3v} \Delta^{\ell^{0, m}-v}, \quad D_v^m := c_4^{m-3\ell^{1, m}+3v-1} c_6 \Delta^{\ell^{1, m}-v}.$$

**Proposition 3.7.** For  $m \in \mathbb{Z}$ ,

$$\mathcal{B}_{TMF}^{0, m} = \{\gamma_0 C_0^m, \gamma_1 C_1^m, \gamma_2 C_2^m, \dots\}, \quad \mathcal{B}_{TMF}^{1, m} = \{\theta_0 D_0^m, \theta_1 D_1^m, \theta_2 D_2^m, \dots\}$$

where

$$\{\gamma_0, \gamma_1, \gamma_2, \dots\} := \begin{cases} \{1, 3, 3, 1, 3, 3, \dots\}, & m \equiv 0, 1, 2 \pmod{9}, \\ \{3, 1, 3, 3, 1, 3, 3, \dots\}, & m \equiv 3, 4, 5 \pmod{9}, \\ \{3, 3, 1, 3, 3, 1, \dots\}, & \text{otherwise} \end{cases}$$

and

$$\{\theta_0, \theta_1, \theta_2, \dots\} := \begin{cases} \{1, 3, 3, 1, 3, 3, \dots\}, & m \equiv 1, 2, 3 \pmod{9}, \\ \{3, 1, 3, 3, 1, 3, 3, \dots\}, & m \equiv 4, 5, 6 \pmod{9}, \\ \{3, 3, 1, 3, 3, 1, \dots\}, & \text{otherwise}. \end{cases}$$

*Proof.* Only the definitions of the sequences  $\{\gamma_0, \gamma_1, \gamma_2, \dots\}$  and  $\{\theta_0, \theta_1, \theta_2, \dots\}$  require justification. By Lemma 3.5(b) and Definition 10, the value of  $\gamma_v$  in  $\gamma_v C_v^m$  depends on whether  $\ell^{0, m} - v$  is divisible by 3. Since  $\ell^{0, m} = \lfloor m/3 \rfloor$ , this 3-divisibility in turn depends on the value of  $m$  modulo 9, as well as the value of  $v$ . An elementary calculation then shows that the above definition of the sequence  $\{\gamma_v\}$  is correct. An analogous argument justifies the above definition of the sequence  $\{\theta_v\}$ .  $\square$

**Corollary 3.8.**  $\pi_0 TMF_{(3)} = \mathbb{Z}_{(3)}\{C_0^0 = 1, 3C_1^0, 3C_2^0, C_3^0, 3C_4^0, 3C_5^0, C_6^0, 3C_7^0, 3C_8^0, \dots\}$ .

*Proof.* This follows from Proposition 3.7 and the fact that  $\pi_0 TMF_{(3)} = {}_{AN}E_\infty^{0,0} TMF_{(3)}$ .  $\square$

**Remark 3.** The enumerations in Definitions 8 and 10 are analogous in terms of how the integer  $m$  compares with the internal degree  $t$ . Specifically,

$$\deg_t(A_v^m) = \deg_t(\gamma_v C_v^m) = 4m, \quad \deg_t(B_v^m) = \deg_t(\theta_v D_v^m) = 4m + 2. \quad (7)$$

## 4 Computation of the maps $g$ and $h$

In this section we initiate our computation of  ${}_{BK}E_2^{*,*}Q(2)_{(3)}$  by computing the kernel and cokernel of the maps  $g : \pi_* TMF_{(3)} \rightarrow \pi_* TMF_{(3)}$  and  $h : B \rightarrow B$  from Definition 6. These computations prove Proposition 2.3.

To begin, we record a useful result in 3-adic analysis.

**Lemma 4.1** ([9], Lemma 8(a)). *If  $0 \neq n \in \mathbb{Z}$ , then  $\nu_3(4^n - 1) = \nu_3(n) + 1$ .*

Next, we compute the effect of  $g$  on an element of  $\pi_*TMF_{(3)}$  (cf. [9], Eq. (21)).

**Lemma 4.2.** *If  $x \in \pi_{2t-s}TMF_{(3)}$  is represented by an element of  $\text{Ext}_\Gamma^{s,t}(B, B)$  in the spectral sequence (6), then*

$$g(x) = (2^t - 1)x.$$

*Proof.* Suppose the representative of  $x$  in  $\text{Ext}_\Gamma^{s,t}(B, B)$  is itself represented in the cobar resolution (see Remark 1) by  $x' \in (\Gamma^{\otimes s})_t$ . Since  $g = \psi_{[2]} - 1$ , the formulas from Definition 3 imply that

$$g : x' \mapsto (2^t - 1)x'.$$

□

**Corollary 4.3.**  $\ker g = \pi_0TMF_{(3)} \oplus (\pi_*TMF_{(3)})_T$ , and

$$\text{coker } g = \pi_*TMF_{(3)} / \sim_{\text{coker } g}$$

where  $\sim_{\text{coker } g}$  denotes the set of all relations of the form  $3^{\nu_3(t)+1} \cdot x = 0$  for  $x$  represented by an element in  $\text{Ext}_\Gamma^{0,t}(B, B)$  with  $t \neq 0$ .

*Proof.* If  $t = 0$ , then  $2^t - 1 = 0$ . By degree counting, the elements in the ANSS for  $TMF_{(3)}$  with  $t$ -degree 0 are exactly those with topological degree  $2t - s = 0$ . Thus  $\pi_0TMF_{(3)} \subset \ker g$  by Lemma 4.2.

Elements in  $(\pi_*TMF_{(3)})_T$  have  $t$ -degree nonzero and even, and they generate copies of  $\mathbb{Z}/(3)$ . Since  $2^t - 1 = 4^{t/2} - 1$  is always divisible by 3 for  $t$  even, Lemma 4.2 implies  $(\pi_*TMF_{(3)})_T \subset \ker g$ .

The remaining elements in  $\pi_*TMF_{(3)}$ , namely the nontrivial elements in  ${}_{AN}E_\infty^{0,t}Q(2)_{(3)}$  for  $t \neq 0$ , are non-torsion, and therefore are not in  $\ker g$ . The formula for  $\ker g$  follows.

The formula for  $\text{coker } g$  follows immediately from Lemmas 4.1 and 4.2. □

**Remark 4.** The relation  $\sim_{\text{coker } g}$  has no effect on elements of  $(\pi_*TMF_{(3)})_T$  since they all generate copies of  $\mathbb{Z}/(3)$ ; it also has no effect on elements of  $\pi_0TMF_{(3)}$  since they are precisely the elements whose representatives in the ANSS lie in  $t$ -degree 0. The affected elements are precisely the remaining ones, namely those whose representatives lie on the 0-line  ${}_{AN}E_\infty^{0,t}TMF_{(3)}$  with  $t \neq 0$ .

The kernel and cokernel of  $h$  were computed by the author in [9].

**Proposition 4.4** ([9], Proposition 3). (a)  $\ker h = \mathbb{Z}_3\{a_{-i,i} : i \geq 1\}$

$$(b) \text{ coker } h = \bigoplus_{i < j \in \mathbb{Z}} (\mathbb{Z}/3^{\nu_3(i+j)+1}\{a_{i,j}\} \oplus \mathbb{Z}/3^{\nu_3(2i+2j+1)+1}\{b_{i,j}\})$$

Corollary 4.3 and Proposition 4.4 together imply Proposition 2.3.

## 5 Computation of $\delta^0$ and $\delta^1$

In this section we compute the kernel and cokernel of the connecting maps  $\delta^0$  and  $\delta^1$ . Using these computations, we prove Theorem 2.4.

## 5.1 $\delta^0 : \ker g \rightarrow \ker h$

By Proposition 2.2, Corollary 4.3, and Proposition 4.4(a), the source of  $\delta^0$  is

$$\ker g = \pi_0 TMF_{(3)} \oplus (\pi_* TMF_{(3)})_T$$

and its target is

$$\ker h = \mathbb{Z}_{(3)}\{a_{-i,i} : i \geq 1\} \subset B_0 = \pi_0 TMF_0(2)_{(3)}.$$

For degree reasons,  $(\pi_* TMF_{(3)})_T \subset \ker \delta^0$ , so it suffices to study the effect of  $\delta^0$  on elements of  $\pi_0 TMF_{(3)}$ .

The computation in ([9], Eq. (24)) implies that for  $v \geq 0$ ,

$$\delta^0(C_v^0) = 2^{8v} \sum_{i=1}^v \left( \binom{3v}{2v+i} 4^{-i} - \binom{3v}{2v-i} 4^i \right) a_{-i,i} - 2^{8v} \sum_{i=v+1}^{2v} \binom{3v}{2v-i} 4^i a_{-i,i}. \quad (8)$$

Thus, with respect to the ordered bases  $\{C_0^0, 3C_1^0, 3C_2^0, C_3^0, 3C_4^0, 3C_5^0, \dots\}$  (see Corollary 3.8) and  $\{a_{-1,1}, a_{-2,2}, \dots\}$ ,

$$\delta^0 \Big|_{\pi_0 TMF_{(3)}} = \begin{bmatrix} 0 & * & * & * \\ \vdots & 3u_1 & * & * \\ & 0 & * & * \\ & \vdots & 3u_2 & * \\ & & 0 & * \\ & & \vdots & u_3 \\ & & & 0 & \ddots \end{bmatrix} \quad (9)$$

where  $u_k = -2^{12k} \in \mathbb{Z}_{(3)}^\times$  for  $k \geq 1$  (cf. [9], Eq. (25)). The form of the matrix in Eq. (9) shows that we may take

$$\{a_{-1,1}, \delta^0(C_1^0), a_{-3,3}, \delta^0(C_2^0), a_{-5,5}, \delta^0(C_3^0), \dots\}$$

as an alternative ordered basis for  $\ker h$ , thereby proving the following proposition.

**Proposition 5.1.**  $\ker \delta^0 = \mathbb{Z}_{(3)}\{1\} \oplus (\pi_* TMF_{(3)})_T$ , and

$$\operatorname{coker} \delta^0 = \mathbb{Z}_{(3)}\{a_{-i,i} : i \geq 1 \text{ odd}\} \oplus \mathbb{Z}/(3)\{\delta^0(C_v^0) : v \geq 1, v \text{ not a multiple of } 3\}.$$

## 5.2 $\delta^1 : \operatorname{coker} g \rightarrow \operatorname{coker} h$

By Proposition 2.2, Corollary 4.3, and Proposition 4.4(b), the source of  $\delta^1$  is

$$\operatorname{coker} g = \pi_* TMF_{(3)} / \sim_{\operatorname{coker} g}$$

and its target is

$$\operatorname{coker} h = \bigoplus_{i < j \in \mathbb{Z}} (\mathbb{Z}/3^{\nu_3(i+j)+1}\{a_{i,j}\} \oplus \mathbb{Z}/3^{\nu_3(2i+2j+1)+1}\{b_{i,j}\}).$$

**Lemma 5.2.** (a)  $(\pi_* TMF_{(3)})_T \subset \ker \delta^1$

(b)  $\ker \delta^1 \Big|_{\pi_0 TMF_{(3)}} = \ker \delta^0 \Big|_{\pi_0 TMF_{(3)}}$ , and similarly for the cokernel.

*Proof.* Part (a) follows from the facts that  $\delta^1$  is induced by  $-\phi_f, \phi_f : r \mapsto 0$  (see Definition 3), and  $\alpha$  (resp.  $\beta$ ) is represented by  $r$  (resp.  $r^2 \otimes r - r \otimes r^2$ ) in the cobar resolution (see Remark 1).

The computation in ([9], Eq. 26) implies that for  $v \geq 0$ ,

$$\delta^1(C_v^0) = \frac{1}{2} \delta^0(C_v^0) \quad (10)$$

which proves (b).  $\square$

By Lemma 3.6, it remains to study  $\delta^1 \Big|_{W^{\epsilon, m}}$  for  $(\epsilon, m) \neq (0, 0)$ , and we shall do so for the remainder of this subsection. First, however, we establish some convenient notational conventions.

**Definition 11.** (a) If  $x = \gamma_v C_v^m \in \mathcal{B}_{TMF}^{0, m}$  (resp.  $y = \theta_v D_v^m \in \mathcal{B}_{TMF}^{1, m}$ ), the  $A_w^m$  term of  $\delta^1(x)$  (resp. the  $B_w^m$  term of  $\delta^1(y)$ ) with the *greatest* subscript  $w$  will be denoted the *leading term*, and we will refer to the remaining terms as *lower order terms* (see Lemma 5.3).

(b) The symbol  $\doteq$  will denote equality up to multiplication by a unit in  $\mathbb{Z}_{(3)}$ .

(c) If  $M$  is a matrix with columns  $M_1, \dots, M_v$  and  $N$  is a matrix with (possibly infinitely many) columns  $N_1, N_2, \dots$ , then  $M \boxplus N$  will denote the matrix with columns

$$M_1, \dots, M_v, N_1, N_2, \dots$$

The following lemma sets up our subsequent computations with  $\delta^1$ .

**Lemma 5.3.** (a) The source of  $\delta^1 \Big|_{W^{0, m}}$  has basis  $\mathcal{B}_{TMF}^{0, m} = \{\gamma_0 C_0^m, \gamma_1 C_1^m, \dots\}$  and its target has basis  $\{A_0^m, A_1^m, \dots\} \subset \text{coker } h$ . Moreover, if  $m < 0$ ,

$$\delta^1(C_v^m) \doteq A_{\lfloor \frac{m-1}{2} \rfloor - 2\ell^{0, m} + 2v}^m + \text{lower order terms} \quad (11)$$

and if  $m > 0$ ,

$$\delta^1(C_v^m) \doteq \begin{cases} A_{\lfloor \frac{m-1}{2} \rfloor - \ell^{0, m} + v}^m + \text{lower order terms}, & 0 \leq v < \ell^{0, m} \\ 0, & v = \ell^{0, m} \\ A_{\lfloor \frac{m-1}{2} \rfloor - 2\ell^{0, m} + 2v}^m + \text{lower order terms}, & v > \ell^{0, m}. \end{cases} \quad (12)$$

(b) The source of  $\delta^1 \Big|_{W^{1,m}}$  has basis  $\mathcal{B}_{TMF}^{1,m} = \{\theta_0 D_0^m, \theta_1 D_1^m, \dots\}$  and its target has basis  $\{B_0^m, B_1^m, \dots\} \subset \text{coker } h$ . Moreover, if  $m \leq 0$ ,

$$\delta^1(D_v^m) \doteq B_{\lfloor \frac{m-1}{2} \rfloor - 2\ell^{1,m} + 2v}^m + \text{lower order terms} \quad (13)$$

and if  $m > 0$ ,

$$\delta^1(D_v^m) \doteq \begin{cases} B_{\lfloor \frac{m-1}{2} \rfloor - \ell^{1,m} + v}^m + \text{lower order terms}, & 0 \leq v < \ell^{1,m} \\ *, & v = \ell^{1,m} \\ B_{\lfloor \frac{m-1}{2} \rfloor - 2\ell^{0,m} + 2v}^m + \text{lower order terms}, & v > \ell^{1,m} \end{cases} \quad (14)$$

where  $*$  = 0 if  $m \neq 13$  (27) and  $*$   $\neq$  0 otherwise.

*Proof.* Equations (11), (12), (13), and (14) follow from the proof of ([9], Proposition 9) and the results cited therein, *mutatis mutandis*. Lemma 3.3, Proposition 3.7, and Remark 3 imply the remaining statements.  $\square$

Guided by Lemma 5.3, we shall now divide our remaining computations of  $\delta^1$  into cases depending on the values of  $\epsilon$  and  $m$ . In each case we will find matrix representations of  $\delta^1$ , just as we did with  $\delta^0$  in Eq. (9).

**Remark 5.** In Eq. (14) when  $m \equiv 13$  (27), the quantity  $*$  =  $\delta^1(D_{\ell^{1,m}}^m)$  is a *nontrivial* linear combination of the generators  $B_v^m$ , which is precisely the reason for the undetermined submodule  $U^* \subset {}_{BK}E_2^{1,*}Q(2)_{(3)}$  in Theorem 1.1. This is explained further below in Case 5. An analogous phenomenon occurred in our computation in [9]: see, e.g., ([9], Remark 2).

*Case 1:*  $\epsilon = 0$ ,  $m < 0$ . By Eq. (11), the matrix representation with respect to the bases given in Lemma 5.3(a) is

$$\delta^1 \Big|_{W^{0,m}} = \begin{bmatrix} \vdots & \vdots & & \\ \gamma_0 u_0 & * & & \\ 0 & * & \vdots & \\ \vdots & \gamma_1 u_1 & * & \\ & 0 & * & \\ & \vdots & \gamma_2 u_2 & \\ & & 0 & \ddots \end{bmatrix} \quad (15)$$

where  $u_0, u_1, u_2, \dots \in \mathbb{Z}_{(3)}^\times$  and  $u_0$  is in the row corresponding to  $A_{\lfloor \frac{m-1}{2} \rfloor - 2\ell^{0,m}}^m$ . Motivated by the structure of this matrix, we may construct an alternative ordered basis for the image, namely

$$\begin{aligned} \{A_0^m, \dots, A_{\lfloor \frac{m-1}{2} \rfloor - 2\ell^{0,m-1}}^m, \delta^1(C_0^m), A_{\lfloor \frac{m-1}{2} \rfloor - 2\ell^{0,m}+1}^m, \\ \delta^1(C_1^m), A_{\lfloor \frac{m-1}{2} \rfloor - 2\ell^{0,m}+3}^m, \\ \delta^1(C_2^m), A_{\lfloor \frac{m-1}{2} \rfloor - 2\ell^{0,m}+5}^m, \dots\}. \end{aligned}$$

From this, we may deduce the kernel and cokernel.

**Lemma 5.4.** For  $m < 0$ ,

$$\ker \delta^1 \Big|_{W^{0,m}} = \begin{cases} \mathbb{Z}/(3)\{3C_1^m, 3C_2^m, 3C_4^m, 3C_5^m, 3C_7^m, 3C_8^m, \dots\}, & m \equiv 0, 1, 2 \pmod{9}, \\ \mathbb{Z}/(3)\{3C_0^m; 3C_2^m, 3C_3^m, 3C_5^m, 3C_6^m, 3C_8^m, 3C_9^m, \dots\}, & m \equiv 3, 4, 5 \pmod{9}, \\ \mathbb{Z}/(3)\{3C_0^m, 3C_1^m, 3C_3^m, 3C_4^m, 3C_6^m, 3C_7^m, \dots\}, & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} \text{coker } \delta^1 \Big|_{W^{0,m}} &= \mathbb{Z}/(3)\{\mathcal{A}\} \oplus \mathbb{Z}/(3^{\nu_3(m)+1}) \left\{ A_0^m, \dots, A_{\lfloor \frac{m-1}{2} \rfloor - 2\ell^{0,m-1}}^m \right\} \\ &\quad \oplus \mathbb{Z}/(3^{\nu_3(m)+1}) \left\{ A_{\lfloor \frac{m-1}{2} \rfloor - 2\ell^{0,m} + i}^m : i \geq 1, \text{ odd} \right\} \end{aligned}$$

where

$$\mathcal{A} := \begin{cases} \{\delta^1(C_1^m), \delta^1(C_2^m), \delta^1(C_4^m), \delta^1(C_5^m), \delta^1(C_7^m), \delta^1(C_8^m), \dots\}, & m \equiv 0, 1, 2 \pmod{9}, \\ \{\delta^1(C_0^m); \delta^1(C_2^m), \delta^1(C_3^m), \delta^1(C_5^m), \delta^1(C_6^m), \delta^1(C_8^m), \delta^1(C_9^m), \dots\}, & m \equiv 3, 4, 5 \pmod{9}, \\ \{\delta^1(C_0^m), \delta^1(C_1^m), \delta^1(C_3^m), \delta^1(C_4^m), \delta^1(C_6^m), \delta^1(C_7^m), \dots\}, & \text{otherwise}. \end{cases}$$

Case 2:  $\epsilon = 0$ ,  $m > 0$ . By Eq. (12), the matrix representation in this case is

$$\delta^1 \Big|_{W^{0,m}} = \begin{bmatrix} \vdots & \vdots & & & & \\ \gamma_0 u_0 & * & & & & \\ 0 & \gamma_1 u_1 & & \vdots & & \\ \vdots & 0 & \ddots & * & \vdots & \\ & \vdots & & \gamma_y u_y & \mathbf{0} & \\ & & & 0 & \mathbf{0} & \\ & & & \vdots & \vdots & \end{bmatrix} \oplus \begin{bmatrix} \vdots & \vdots & & & & \\ \gamma_{y+2} u_{y+2} & * & & & & \\ 0 & * & & \vdots & & \\ \vdots & \gamma_{y+3} u_{y+3} & & * & & \\ & 0 & & * & & \\ & \vdots & & \gamma_{y+4} u_{y+4} & & \\ & & & 0 & \ddots & \end{bmatrix} \quad (16)$$

where the  $u_i$  are units in  $\mathbb{Z}_{(3)}$ . Here,  $u_0$  is in the row corresponding to  $A_{\lfloor \frac{m-1}{2} \rfloor - \ell^{0,m}}^m$ ,  $u_y$  is in the row corresponding to  $A_{\lfloor \frac{m-1}{2} \rfloor - 1}^m$ ,  $u_{y+2}$  is in the row corresponding to  $A_{\lfloor \frac{m-1}{2} \rfloor + 2}^m$ , and the zero column in bold corresponds to  $C_{\ell^{0,m}}^m$ . As in the previous case, we may deduce the kernel and cokernel by constructing an alternative basis for the image; this time it is

$$\begin{aligned} \{A_0^m, \dots, A_{\lfloor \frac{m-1}{2} \rfloor - \ell^{0,m-1}}^m; \delta^1(C_0^m), \dots, \delta^1(C_{\ell^{0,m-1}}^m); A_{\lfloor \frac{m-1}{2} \rfloor}; A_{\lfloor \frac{m-1}{2} \rfloor + 1}, \delta^1(C_{\ell^{0,m+1}}^m), \\ A_{\lfloor \frac{m-1}{2} \rfloor + 3}, \delta^1(C_{\ell^{0,m+2}}^m), \\ A_{\lfloor \frac{m-1}{2} \rfloor + 5}, \delta^1(C_{\ell^{0,m+3}}^m), \dots\}. \end{aligned}$$

**Lemma 5.5.** For  $m < 0$ ,  $\ker \delta^1 \Big|_{W^{0,m}} = \mathbb{Z}/(3^{\nu_3(m)+1})\{C_{\ell^{0,m}}^m\} \oplus K$  where

$$K := \begin{cases} \mathbb{Z}/(3)\{3C_1^m, 3C_2^m, 3C_4^m, 3C_5^m, 3C_7^m, 3C_8^m, \dots\}, & m \equiv 0, 1, 2 \pmod{9}, \\ \mathbb{Z}/(3)\{3C_0^m; 3C_2^m, 3C_3^m, 3C_5^m, 3C_6^m, 3C_8^m, 3C_9^m, \dots\}, & m \equiv 3, 4, 5 \pmod{9}, \\ \mathbb{Z}/(3)\{3C_0^m, 3C_1^m, 3C_3^m, 3C_4^m, 3C_6^m, 3C_7^m, \dots\}, & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} \text{coker } \delta^1 \Big|_{W^{0,m}} &= \mathbb{Z}/(3)\{\mathcal{A}\} \oplus \mathbb{Z}/(3^{\nu_3(m)+1}) \left\{ A_0^m, \dots, A_{\lfloor \frac{m-1}{2} \rfloor - \ell^{0,m-1}}^m; A_{\lfloor \frac{m-1}{2} \rfloor}^m \right\} \\ &\quad \oplus \mathbb{Z}/(3^{\nu_3(m)+1}) \left\{ A_{\lfloor \frac{m-1}{2} \rfloor + i}^m : i \geq 1, \text{ odd} \right\} \end{aligned}$$

where  $\mathcal{A}$  is defined as in Lemma 5.4.

Case 3:  $\epsilon = 1$ ,  $m \leq 0$ . By Eq. (13), the matrix representation with respect to the bases given in Lemma 5.3(b) is  $\delta^1 \Big|_{W_{1,m}}$  identical in form to the one in (15), except with  $\gamma_i$  replaced by  $\theta_i$  everywhere. In this case the unit  $u_0$  appears in the row corresponding to  $B_{\lfloor \frac{m-1}{2} \rfloor - 2\ell^{1,m}}^m$ . Therefore, we may argue as in Case 1 to obtain the following lemma.

**Lemma 5.6.** For  $m \leq 0$ ,

$$\ker \delta^1 \Big|_{W^{1,m}} = \begin{cases} \mathbb{Z}/(3)\{3D_1^m, 3D_2^m, 3D_4^m, 3D_5^m, 3D_7^m, 3D_8^m, \dots\}, & m \equiv 1, 2, 3 \pmod{9}, \\ \mathbb{Z}/(3)\{3D_0^m; 3D_2^m, 3D_3^m, 3D_5^m, 3D_6^m, 3D_8^m, 3D_9^m, \dots\}, & m \equiv 4, 5, 6 \pmod{9}, \\ \mathbb{Z}/(3)\{3D_0^m, 3D_1^m, 3D_3^m, 3D_4^m, 3D_6^m, 3D_7^m, \dots\}, & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} \text{coker } \delta^1 \Big|_{W^{1,m}} &= \mathbb{Z}/(3)\{\mathcal{B}\} \oplus \mathbb{Z}/(3^{\nu_3(2m+1)+1}) \left\{ B_0^m, \dots, B_{\lfloor \frac{m-1}{2} \rfloor - 2\ell^{0,m-1}}^m \right\} \\ &\quad \oplus \mathbb{Z}/(3^{\nu_3(2m+1)+1}) \left\{ B_{\lfloor \frac{m-1}{2} \rfloor - 2\ell^{0,m} + i}^m : i \geq 1, \text{ odd} \right\} \end{aligned}$$

where

$$\mathcal{B} := \begin{cases} \{\delta^1(D_1^m), \delta^1(D_2^m), \delta^1(D_4^m), \delta^1(D_5^m), \delta^1(D_7^m), \delta^1(D_8^m), \dots\}, & m \equiv 1, 2, 3 \pmod{9}, \\ \{\delta^1(D_0^m); \delta^1(D_2^m), \delta^1(D_3^m), \delta^1(D_5^m), \delta^1(D_6^m), \delta^1(D_8^m), \delta^1(D_9^m), \dots\}, & m \equiv 4, 5, 6 \pmod{9}, \\ \{\delta^1(D_0^m), \delta^1(D_1^m), \delta^1(D_3^m), \delta^1(D_4^m), \delta^1(D_6^m), \delta^1(D_7^m), \dots\}, & \text{otherwise.} \end{cases}$$

Case 4:  $\epsilon = 1$ ,  $m > 0$ ,  $m \neq 13 \pmod{27}$ . By Eq. (14), the matrix representation in this case is identical in form to the one in (16), except with  $\gamma_i$  replaced by  $\theta_i$  everywhere. Moreover,  $u_0$  is in the row corresponding to  $B_{\lfloor \frac{m-1}{2} \rfloor - \ell^{1,m}}^m$ ,  $u_y$  is in the row corresponding to  $B_{\lfloor \frac{m-1}{2} \rfloor - 1}^m$ , and  $u_{y+2}$  is in the row corresponding to  $B_{\lfloor \frac{m-1}{2} \rfloor + 2}^m$ . Therefore, we may argue as in Case 2 to obtain the following lemma.

**Lemma 5.7.** For  $m > 0$  and  $m \neq 13 \pmod{27}$ ,  $\ker \delta^1 \Big|_{W^{1,m}} = \mathbb{Z}/(3^{\nu_3(2m+1)+1})\{D_{\ell^{1,m}}^m\} \oplus K'$  where

$$K' := \begin{cases} \mathbb{Z}/(3)\{3D_1^m, 3D_2^m, 3D_4^m, 3D_5^m, 3D_7^m, 3D_8^m, \dots\}, & m \equiv 1, 2, 3 \pmod{9}, \\ \mathbb{Z}/(3)\{3D_0^m; 3D_2^m, 3D_3^m, 3D_5^m, 3D_6^m, 3D_8^m, 3D_9^m, \dots\}, & m \equiv 4, 5, 6 \pmod{9}, \\ \mathbb{Z}/(3)\{3D_0^m, 3D_1^m, 3D_3^m, 3D_4^m, 3D_6^m, 3D_7^m, \dots\}, & \text{otherwise} \end{cases}$$



and

$$\begin{aligned} \operatorname{coker} \delta^1 \Big|_{W^{1,m}} &= \mathbb{Z}/(3)\{\mathcal{B}\} \oplus \mathbb{Z}/(3^{\nu_3(2m+1)+1}) \left\{ B_0^m, \dots, B_{\lfloor \frac{m-1}{2} \rfloor - \ell^{0,m-1}}^m; B_{\lfloor \frac{m-1}{2} \rfloor}^m \right\} \\ &\quad \oplus \mathbb{Z}/(3^{\nu_3(2m+1)+1}) \left\{ B_{\lfloor \frac{m-1}{2} \rfloor + i}^m : i \geq 1, \text{ odd} \right\} \end{aligned}$$

where  $\mathcal{B}$  is defined as in Lemma 5.6.

Case 5:  $\epsilon = 1$ ,  $m > 0$ ,  $m \equiv 13 \pmod{27}$ . In this final case, Eq. (14) implies that the matrix representation is identical in form to the one in Case 4, except that the column in bold is no longer a column of zeros. Rather, by ([9], Lemma 12(a)) it has at least one nonzero entry in and above the row containing  $u_y$ . This yields the following lemma.

**Lemma 5.8.** *Suppose  $m > 0$  and  $m \equiv 13 \pmod{27}$ .*

(a) *The cokernel of  $\delta^1 \Big|_{W^{1,m}}$  has the same presentation as in Case 4, but now including a single nontrivial relation*

$$\delta^1(D_{\ell^{1,m}}^m) = 0 \tag{17}$$

*among its generators.*

(b) *The kernel of  $\delta^1 \Big|_{W^{1,m}}$  has, as a direct summand,*

$$K'' := \mathbb{Z}/(3)\{3D_{\ell^{1,m}+1}^m, 3D_{\ell^{1,m}+2}^m, 3D_{\ell^{1,m}+4}^m, 3D_{\ell^{1,m}+5}^m, 3D_{\ell^{1,m}+7}^m, 3D_{\ell^{1,m}+8}^m, \dots\}.$$

**Definition 12.** The graded module  $U^*$  in Theorem 1.1 is determined by the direct sum decomposition

$$\ker \delta^1 \Big|_{W^{1,m}} = K'' \oplus U^{4m+2}.$$

We do not have sufficient knowledge of the coefficients in the columns of the matrix in Eq. (16) corresponding to  $D_0^m, \dots, D_{\ell^{1,m}}^m$  (e.g., their 3-divisibility) to explicitly compute  $U$ .

*Proof of Theorem 2.4.* Propositions 2.2 and 5.1 yield the result for  $H^0\mathcal{G}_*^\bullet$ . Proposition 5.1 also implies that  $\operatorname{coker} \delta^0$  is concentrated in  $t$ -degree zero, while Proposition 5.2 implies that  $\ker \delta^1$  is a free  $\mathbb{Z}_{(3)}$ -module of rank 1 generated by  $1_{\pi_*TMF_{(3)}}$ . Thus, the short exact sequence (4) splits as claimed in Proposition 2.2, which means the result for  $H^1C^*$  follows from Proposition 5.1 combined with Lemmas 5.4, 5.5, 5.6, 5.7, and 5.8(b). Finally, the result for  $H^2C^*$  follows from Proposition 2.2, Lemmas 5.4 – 5.7 and 5.8(a), and Definition 12.  $\square$

## 6 Higher differentials in the BKSS

In this final section we briefly examine the anatomy of the differentials on the  $E_r$ -page of the BKSS for  $Q(2)_{(3)}$  for  $r \geq 2$ .

A schematic diagram of  ${}_{BK}E_2^{s,t}Q(2)_{(3)}$  is shown in Figure 2, with  $s$  along the vertical axis and  $t - s$  along the horizontal axis. Included in the diagram is an example of a possibly

nontrivial  $d_2$ -differential  ${}_{BK}E_2^{0,8}Q(2)_{(3)} \rightarrow {}_{BK}E_2^{2,7}Q(2)_{(3)}$ . In fact, since the only nontrivial rows in the diagram are those corresponding to  $s = 0$ ,  $s = 1$ , and  $s = 2$  by Theorem 1.1, the only possibly nontrivial  $d_2$ -differentials are those that map from the 0-line to the 2-line. Moreover, by sparseness, all  $d_2$ -differentials mapping into or out of the 1-line are identically

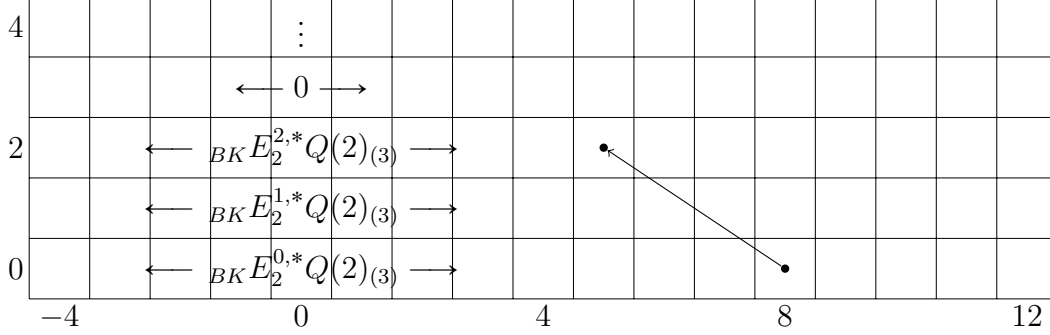


Figure 2: The  $E_2$ -term of the BKSS for  $Q(2)_{(3)}$ .

zero, as are all  $d_r$ -differentials for  $r \geq 3$ . The following theorem summarizes the situation.

**Theorem 6.1.** *In the BKSS for  $Q(2)_{(3)}$ ,*

(a) *The only possibly nontrivial  $d_2$ -differentials have the form*

$$d_2 : {}_{BK}E_2^{0,t}Q(2)_{(3)} \rightarrow {}_{BK}E_2^{2,t-1}Q(2)_{(3)},$$

(b) *All elements in  ${}_{BK}E_2^{1,*}Q(2)_{(3)}$  are permanent cycles,*

(c) *The spectral sequence collapses at  $E_3$ , i.e.,  ${}_{BK}E_3^{s,t}Q(2)_{(3)} = {}_{BK}E_\infty^{s,t}Q(2)_{(3)}$ .*

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